# Quantum Three Wave Interaction Models : Bethe Anstz and Statistical Mechanics(Development of Soliton Theory)



Ohkuma, Kenji

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# Quantum Three Wave Interaction Models: Bethe Anstz and Statistical Mechanics

東大・教養　大熊建司 ( Kenji Ohkuma )

## §1. Introduction

The quantum three wave interaction (Q3WI, hereafter) model has applications in many fields of physics, for example, nonlinear optics, plasma physics, solid state physics, etc.[1,2] In the present paper we construct Bethe states for three choices of statistics. The thermodynamics for one case of them is studied. In this case there does not exist any bound state. Imposing the periodic boundary conditions, two integral equations for the densities of states for particles and holes are obtained. Further giving a form of an entropy and minimizing the free energy under a condition of fixed particle densities, integral equations for the thermal equilibrium are derived.

The construction of this paper is as follows. In the following section, we introduce the model and show its Bethe states. In §3, we study the thermal equilibrium from periodic boundary condition and the condition for the minimal free energy. In §4, three limiting cases; $g \to 0$, $g \to \infty$ and $T \to 0$, are studied. The final section is devoted to the concluding remarks.

## §2. The Model and the Bethe States

The quantum three wave interaction (Q3WI, hereafter) model in 1-dimensional space is given by the Hamiltonian;

$$H = \int dx \{ \sum_{j=1}^{3} c_j Q_j^*(x) \frac{1}{i} \frac{\partial}{\partial x} Q_j(x)$$

$$+ g[Q_2^*(x) Q_3(x) Q_1(x) + Q_1^*(x) Q_3^*(x) Q_2(x)] \}, \qquad (2.1)$$

where c's are distinct constant velocities, g is the coupling constant, and $Q^*$'s and Q's are creation and annihilation operators, respectively. Eq.(2.1) suggests that the number of each particle is not conserved. Three choices of statistics can be considered;

1) Boson model  All fields are bosons, i.e.

$$[Q_j(x,t),Q_k^*(y,t)] = \delta_{jk}\delta(x-y), \quad j,k = 1,2,3, \quad \text{etc.}$$

2) Fermion model I  The fields $Q_1$ and $Q_3$ are fermions and the field $Q_2$ is a boson, i.e.

$$\{Q_j(x,t),Q_k^*(y,t)\} = \delta_{jk}\delta(x-y), \qquad j,k = 1,3,$$

$$[Q_2(x,t),Q_k^*(y,t)] = \delta_{2k}\delta(x-y), \qquad k = 1,2,3, \quad \text{etc.}$$

field $Q_2$ is a boson.

3) Fermion model II  The fields $Q_1$ and $Q_2$ are fermions and the field $Q_3$ is a boson, i.e.

$$\{Q_j(x,t),Q_k^*(y,t)\} = \delta_{jk}\delta(x-y), \qquad j,k = 1,2, \quad \text{etc.}$$

For the description of eigenstate we prepare some notations.[3]

First we define the vacuum state $|0>$ as

$$Q_j(x,t)|0> = 0, \qquad j=1,2,3. \tag{2.2}$$

Ket states created by only one kind of field operators are expressed as

$$|\lambda_1,\cdots,\lambda_N> = \int\cdots\int dx_1\cdots dx_N\theta(x_1>\cdots>x_N)$$

$$\times \exp[i(p_1x_1+\cdots+p_Nx_N)]Q_1^*(x_1)\cdots Q_1^*(x_N)|0>, \tag{2.3a}$$

$$|\mu_1,\cdots,\mu_N> = \int\cdots\int dx_1\cdots dx_N\theta(x_1>\cdots>x_N)$$

$$\times \exp[i(q_1x_1+\cdots+q_Nx_N)]Q_3^*(x_1)\cdots Q_3^*(x_N)|0>, \tag{2.3b}$$

$$|\lambda_1+\mu_1,\cdots,\lambda_N+\mu_N> = \int\cdots\int dx_1\cdots dx_N\theta(x_1>\cdots>x_N)$$

$$\times \exp\{i[(p_1+q_1)x_1+\cdots+(p_N+q_N)x_N]Q_2^*(x_2)\cdots Q_2^*(x_N)|0>, \tag{2.3c}$$

with

$$\theta(x_1>\cdots>x_N) = \theta(x_1-x_2)\theta(x_2-x_3)\cdots\theta(x_{N-1}-x_N),$$

$$\theta(x) = \begin{cases} 1 & \text{for } x > 0 \\ 1/2 & \text{for } x = 0 \\ 0 & \text{for } x < 0, \end{cases} \tag{2.4}$$

$$p_j = (c_2-c_3)\lambda_j, \quad q_j = (c_1-c_2)\mu_j. \tag{2.5}$$

Argument $\lambda_j$ ($\mu_j$) means a $Q_1$- ($Q_3$-) particle with a wave number $p_j$ ($q_j$). Similarly $\lambda_j+\mu_j$ means a $Q_2$-particle with a wave number $p_j+q_k$. Ket states with more than one kind of particles are expressed in a similar way, e.g.

$$|\lambda_1,\lambda_2+\mu_1,\mu_2> = \iiint dx_1dx_2dx_3\theta(x_1>x_2>x_3)$$

$$\times \exp\{i[p_1x_1+(p_2+q_1)x_2+q_2x_3]\}Q_1^*(x_1)Q_2^*(x_2)Q_3^*(x_3)|0>. \tag{2.3d}$$

The Hamiltonian commutes with the number operators;

$$\hat{M} = \int dx(Q_1^* Q_1 + Q_2^* Q_2), \qquad \hat{N} = \int dx(Q_2^* Q_2 + Q_3^* Q_3). \tag{2.6}$$

Eigenvalues for these operators are non-negative integers. Therefore, we use them to classify eigenstates. The eigenstate for $\hat{M}$ and $\hat{N}$ is expressed as $||M,N\gg$, i.e.

$$\hat{M}||M,N\gg = M||M,N\gg, \qquad \hat{N}||M,N\gg = N||M,N\gg, \tag{2.7}$$

where M and N are eigenvalues for $\hat{M}$ and $\hat{N}$, respectively. The terms in the state $||M,N\gg$ are classified into $[\min(M,N)+1]$ kinds according to the number of $Q_2^*$-operators. The terms with $\ell Q_2^*$'s has $(M-\ell)Q_1^*$'s, and $(N-\ell)Q_3^*$'s, and $\ell$ satisfies the condition;

$$0 \leqq \ell \leqq \min(M,N). \tag{2.8}$$

The eigenstate is determined up to a constant factor by giving the quantum numbers M, N and the set of the wave numbers $\{p_1,\cdots,p_M,q_1,\cdots,q_N\}$ with eq.(2.5). We assume that $MN\neq 0$ hereafter. We can write the state $||M,N\gg$ in the case $M\geqq N$ as follows

$$||M,N\gg \; = \; [\lambda_1,\cdots,\lambda_M,\mu_1,\cdots,\mu_N]|\lambda_1,\cdots,\lambda_M,\mu_1,\cdots,\mu_N>$$

$$+ [\lambda_1,\cdots,\mu_N,\mu_{N-1}]|\lambda_1,\cdots,\mu_N,\mu_{N-1}> + \cdots$$

$$+ [\mu_N,\cdots,\mu_1,\lambda_M,\cdots,\lambda_1]|\mu_N,\cdots,\mu_1,\lambda_M,\cdots,\lambda_1>$$

$$+ [\lambda_1,\cdots,\lambda_M+\mu_1,\cdots,\mu_N]|\lambda_1,\cdots,\lambda_M+\mu_1,\cdots,\mu_N>$$

$$+ \cdots$$

$$+[\lambda_1+\mu_1,\cdots,\lambda_N+\mu_N,\lambda_{N+1},\cdots,\lambda_M]|\lambda_1+\mu_1,\cdots,\lambda_N+\mu_N,\lambda_{N+1},\cdots,\lambda_M>$$

$$+ \cdots. \tag{2.9}$$

To be a Bethe state, the above coefficients (square brackets) are related by the following rules;[5)]

$$[\cdots,\mu_k,\lambda_j,\cdots] = S_1(\lambda_j-\mu_k)[\cdots,\lambda_j,\mu_k,\cdots],$$

$$[\cdots,\lambda_k,\lambda_j,\cdots] = S_2(\lambda_j-\lambda_k)[\cdots,\lambda_j,\lambda_k,\cdots],$$

$$[\cdots,\mu_k,\mu_j,\cdots] = S_3(\mu_j-\mu_k)[\cdots,\mu_j,\mu_k,\cdots],$$

$$[\cdots,\lambda_j+\mu_k,\cdots] = S_+(\lambda_j-\mu_k)[\cdots,\lambda_j,\mu_k,\cdots], \tag{2.10}$$

where $S_1, S_2, S_3, S_+$ are expressed for each choice of statistics as follows;

1) Boson model

$$S_1(\nu) = (\nu-i\kappa)/(\nu+i\kappa), \quad S_2(\nu) = S_3(\nu) = (\nu+2i\kappa)/(\nu-2i\kappa),$$

$$S_+(\nu) = -2\kappa(c_1-c_3)/g(\nu+i\kappa), \tag{2.11}$$

$$\kappa = g^2/2(c_1-c_2)(c_2-c_3)(c_3-c_1). \tag{2.12}$$

The same result has already been obtained by Kulish and Reshetikhin through algebraic Bethe Ansatz in this case.[7)]

2) Fermion model I

$$S_1(\nu) = -(\nu-i\kappa)/(\nu+i\kappa), \quad S_2(\nu) = S_3(\nu) = -1,$$

$$S_+(\nu) = -2\kappa(c_1-c_3)/g(\nu+i\kappa). \tag{2.13}$$

3) Fermion model II

$$S_1(\nu) = (\nu-i\kappa)/(\nu+i\kappa), \quad S_2(\nu) = -1,$$

$$S_3(\nu) = (\nu+2i\kappa)/(\nu-2i\kappa), \quad S_+(\nu) = -2\kappa(c_1-c_3)\,/g(\nu+i\kappa). \tag{2.14}$$

As the number of each particle is not conserved, we have to consider $S_+$, which does not appear in usual Bethe states.

For all models, the energy eigenvalue is

$$E = c_1(p_1+\cdots+p_M) + c_3(q_1+\cdots+q_N). \tag{2.15}$$

The number of terms with $\ell Q_2^*$'s is $(M+N-\ell)!\,{}_MP_\ell\cdot{}_NC_\ell$ for

$0 \leq \ell \leq \min(M,N)$.

Next we show the condition where bound states in the eigenstates occur.[3)]

1) Boson model

(1) Bound states of $Q_1$-particles occur when

$$(c_1-c_2)(c_3-c_1) < 0. \tag{2.16}$$

(2) Bound states of $Q_3$-particles occur when

$$(c_2-c_3)(c_3-c_1) < 0. \tag{2.17}$$

For distinct c's, at least one of eqs.(2.16) and (2.17) is always satisfied, which means, there can always exist bound states in the Boson model.

2) Fermion model I

In this case, no bound states occur.

3) Fermion model II

(1) The bound state of $Q_1$-particles does not occur.

(2) The bound state of $Q_3$-particles occurs when

$$(c_2-c_3)(c_3-c_1) < 0. \tag{2.18}$$

## §3. Periodic Boundary Conditions

We consider the Fermion model I, where the fields $Q_1$ and $Q_3$ are fermions and the field $Q_2$ is a boson. From eqs.(2.9) and (2.13) the eigenstate $||M,N>>$ can be expressed as follows;

$$||M,N>> = \alpha\int\cdots\int dx_1\cdots dx_M dy_1\cdots dy_N \Psi(x_1,\cdots,x_M,y_1,\cdots,y_N)$$
$$\times\, Q_1^*(x_1)\cdots Q_1^*(x_M)Q_3^*(y_1)\cdots Q_3^*(y_N)|0>$$
$$+\sum_{\ell=1}^{\min(M,N)}\{\text{ terms with } \ell Q_2^*\text{-operators }\}, \qquad (3.1)$$

where

$$\Psi(x_1,\cdots,y_N) = \exp[i(p_1x_1+\cdots p_Mx_M+q_1y_1+\cdots+q_Ny_N)]$$
$$\times\{\ \theta(x_1>\cdots>x_M>y_1>\cdots>y_N)$$
$$+\theta(x_1>\cdots>x_M>y_1>\cdots>y_N>y_{N-1}) + \cdots$$
$$-S_1(\lambda_M-\mu_1)\theta(x_1>\cdots>y_1>x_M>\cdots>y_N) + \cdots$$
$$+\prod_{j,k}[-S_1(\lambda_j-\mu_k)]\theta(y_N>\cdots>y_1>x_M>\cdots>x_1)\ \}. \qquad (3.2)$$

When we set a volume of the system L, then the periodic boundary condition for the terms without $Q_2^*$-operators is

$$\Psi(x_j=L) = \Psi(x_j=0), \quad \Psi(y_j=L) = \Psi(y_j=0), \qquad (3.3)$$

i.e.

$$\exp(ip_jL) = (-1)^N \exp[i\sum_{\ell=1}^{N}\phi(\lambda_j-\mu_\ell)],$$

$$\exp(iq_jL) = (-1)^M \exp[i\sum_{\ell=1}^{M}\phi(\mu_j-\lambda_\ell)], \tag{3.4}$$

where

$$\phi(\nu) = -i\ell n[S_1(\nu)] = 2\tan^{-1}(\nu/\kappa), \quad -\pi<\phi<\pi. \tag{3.5}$$

It is easily shown that periodic boundary conditions for the terms with $Q_2$-particles are automatically satisfied, when the conditions (3.3) are assumed. So it is enough for us to consider conditions only for $Q_1$- and $Q_3$-fields. Take the logarithm of eq.(2.3), we get

$$p_jL = 2\pi I_j+\sum_{\ell=1}^{N}\phi(\lambda_j-\mu_\ell), \qquad q_jL = 2\pi J_j+\sum_{\ell=1}^{M}\phi(\mu_j-\lambda_\ell), \tag{3.6}$$

where

$I_j$ is an integer (a half integer) when N is even (odd),

$J_j$ is an integer (a half integer) when M is even (odd).

Here we define the functions $h_1$ and $h_3$ as

$$Lh_1 = p_jL - \sum_{\ell=1}^{N}\phi(\lambda_j-\mu_\ell),$$

$$Lh_3 = q_jL - \sum_{\ell=1}^{M}\phi(\mu_j-\lambda_\ell). \tag{3.7}$$

These functions become continuous monotonic functions in large volume limit.

We consider integers (or half integers) I's and K's which satisfy

$$I_j \in Lh_1(p_j)/2\pi, \quad K_j \notin Lh_1(p_j)/2\pi. \qquad (3.8)$$

Here we impose the following restriction $I_j$, $K_j \geqq I_{min}$. $I_{min}$ corresponds to the cut-off wave number $p_K$ which will appear soon. I's correspond to $Q_1$-particles, and K's correspond to $Q_1$-holes. The density of states $\rho_1$ and $\rho_1^h$ are defined in the large volume limit as follows;

$$L\rho_1(p)dp = \{ \text{ no. of I's in } (p,p+dp) \},$$

$$L\rho_1^h(p)dp = \{ \text{ no. of K's in } (p,p+dp) \}. \qquad (3.9)$$

Thus,

$$dh_1(p)/dp = 2\pi[\rho_1(p)+\rho_1^h(p)] = 2\pi f_1(p). \qquad (3.10a)$$

For $Q_3$- particles and holes, the same treatment is possible and we get

$$dh_3(q)/dq = 2\pi[\rho_3(q)+\rho_3^h(q)] = 2\pi f_3(q). \qquad (3.10b)$$

The last equality in (3.10) means definition of $f_j$. In the large volume limit eq.(3.7) becomes

$$h_1(p) = p-\int_{q_K}^{\infty} dq\ \rho_3(q)\phi(\lambda_p-\mu_q),$$

$$h_3(q) = q-\int_{p_K}^{\infty} dp\ \rho_1(p)\phi(\mu_q-\lambda_p). \tag{3.11}$$

Here $p_K$ and $q_K$ are cut-off wave number for $Q_1$- and $Q_3$-particles, respectively. Differentiate these equalities and use eq.(3.11), we have

$$2\pi f_1(p) = 2\pi[\rho_1(p)+\rho_1^h(p)]$$

$$= 1-\frac{2\kappa}{c_2-c_3}\int_{q_K}^{\infty} K(p,q)\rho_3(q)dq,$$

$$2\pi f_3(q) = 2\pi[\rho_3(q)+\rho_3^h(q)]$$

$$= 1-\frac{2\kappa}{c_1-c_2}\int_{p_K}^{\infty} K(p,q)\rho_1(p)dp. \tag{3.12}$$

where

$$K(p,q) = [(\frac{p}{c_2-c_3}-\frac{q}{c_1-c_2})^2+\kappa^2]^{-1}. \tag{3.13}$$

With $\rho_1$ and $\rho_3$, the energies per particle are expressed as

$$E_1/M = D_1^{-1}\int_{p_K}^{\infty} dp\ c_1 p\rho_1(p),$$

$$E_3/N = D_3^{-1}\int_{q_K}^{\infty} dq\ c_3 q\rho_3(q), \tag{3.14}$$

where $D_1$ and $D_3$ are particle densities of $Q_1$ and $Q_3$ per unit volume, i.e. $D_1=M/L$ $D_3=N/L$, respectively.

Next we consider the free energy of the state. Along the disscussion of Yang and Yang, the entropy of the $Q_1$- and $Q_3$-fields are[4)]

$$S_1 = L\int_{p_K}^{\infty} dp\{[\rho_1(p)+\rho_1^h(p)]\ell n[\rho_1(p)+\rho_1^h(p)]$$

$$-\rho_1(p)\ell n\rho_1(p)-\rho_1^h(p)\ell n\rho_1^h(p)\},$$

$$S_3 = L\int_{q_K}^{\infty} dq\{[\rho_3(q)+\rho_3^h(q)]\ell n[\rho_3(q)+\rho_3^h(q)]$$

$$-\rho_3(q)\ell n\rho_3(q)-\rho_3^h(q)\ell n\rho_3^h(q)\}. \tag{3.15}$$

Then the free energy is

$$F = E_1+E_3-T(S_1+S_3). \tag{3.16}$$

Minimize the free energy under the condition (3.13), i.e. take the variation of

$$F+A_1T[M-L\int_{p_K}^{\infty} dp\ \rho_1(p)]+A_3T[N-L\int_{q_K}^{\infty} dq\ \rho_3(q)] \tag{3.17}$$

then set it zero, we get

$$-A_1T+c_1p+T\ell n[\frac{\rho_1}{\rho_1^h}(p)]+\frac{\kappa T}{\pi(c_1-c_2)}\int_{q_K}^{\infty} dq\ K(p,q)\ell n[1+\frac{\rho_3}{\rho_3^h}(q)] = 0,$$

$$-A_3T+c_3q+T\ell n[\frac{\rho_3}{\rho_3^h}(q)]+\frac{\kappa T}{\pi(c_2-c_3)}\int_{p_K}^{\infty} dp\ K(p,q)\ell n[1+\frac{\rho_1}{\rho_1^h}(p)] = 0, \tag{3.18}$$

where $A_1$ and $A_3$ are the Lagrange multipliers for the condition (3.14). Here we define $\varepsilon_1$ and $\varepsilon_3$ as

$$\exp[-\varepsilon_1(p)/T] = \rho_1(p)/\rho_1^h(p),$$

$$\exp[-\varepsilon_3(q)/T] = \rho_3(q)/\rho_3^h(q). \tag{3.19}$$

Use these $\varepsilon$'s, eq.(3.18) becomes

$$\varepsilon_1(p) = -A_1T+c_1p+\frac{\kappa T}{\pi(c_1-c_2)}\int_{q_K}^{\infty} dq\ K(p,q)\ell n\{1+\exp[-\varepsilon_3(q)/T]\},$$

$$\varepsilon_3(q) = -A_3T+c_3q+\frac{\kappa T}{\pi(c_2-c_3)}\int_{p_K}^{\infty} dp\ K(p,q)\ell n\{1+\exp[-\varepsilon_1(p)/T]\}. \tag{3.20}$$

Equation (3.12) becomes

$$2\pi f_1(p) = 1-\frac{2\kappa}{c_2-c_3}\int_{q_K}^{\infty} dq\ K(p,q)f_3(q)/\{1+\exp[\varepsilon_3(q)/T]\},$$

$$2\pi f_3(q) = 1-\frac{2\kappa}{c_1-c_2}\int_{p_K}^{\infty} dp\ K(p,q)f_1(p)/\{1+\exp[\varepsilon_1(p)/T]\}. \tag{3.21}$$

## §4. Special Cases

In this section we consider three limits; strong coupling limit ($g\to\infty$), weak coupling limit ($g\to 0$) and zero temperature limit ($T\to 0$).

### §§4.1 Strong coupling limit: $g \to \infty$

In this limit the integals in (3.20) and (3.21) vanish. Thus,

$$\varepsilon_1(p) = -A_1+c_1p, \quad \varepsilon_3(q) = -A_3+c_3q,$$

$$2\pi\rho_1(p) = z_1\exp(-c_1p/T)[1+z_1\exp(-c_1p/T)]^{-1},$$

$$2\pi\rho_1^h(p) = [1+z_1\exp(-c_1p/T)]^{-1},$$

$$2\pi\rho_3(q) = z_3\exp(-c_3q/T)[1+z_3\exp(-c_3q/T)]^{-1},$$

$$2\pi\rho_3^h(q) = [1+z_3\exp(-c_3q/T)]^{-1}, \tag{4.1}$$

where

$$z_1 = \exp A_1/T, \quad z_3 = \exp A_3/T. \tag{4.2}$$

These results show that the particles behave like free fermion gases. This can be understood easily, because in this limit $S_+$ becomes zero, which means that $Q_2$-particles cannot exist and $Q_1$- and $Q_3$-particles do not interact each other.

## §§4.2 Weak coupling limit: $g \to 0$

As $g \to 0$,

$$\kappa K(p,q) \to -\pi\delta[p/(c_2-c_3)-q/(c_1-c_2)]. \tag{4.3}$$

Thus, eq.(3.20) becomes

$$\varepsilon_1(p) = -A_1+c_1p-T\ell n\{1+\exp[-\varepsilon_3(\frac{c_1-c_2}{c_2-c_3}p)/T]\},$$

$$\varepsilon_3(q) = -A_3+c_3q-T\ell n\{1+\exp[-\varepsilon_1(\frac{c_2-c_3}{c_1-c_2}q)/T]\}. \tag{4.4}$$

Equations (4.3) and (3.21) gives

$$2\pi f_1(p) = 2\pi[\rho_1(p)+\rho_1^h(p)] = 1+2\pi\frac{c_1-c_2}{c_2-c_3}\rho_3(\frac{c_1-c_2}{c_2-c_3}p),$$

$$2\pi f_3(q) = 2\pi[\rho_3(q)+\rho_3^h(q)] = 1+2\pi\frac{c_2-c_3}{c_1-c_2}\rho_1(\frac{c_2-c_3}{c_1-c_2}q). \tag{4.5}$$

Thus,

$$\begin{aligned}2\pi\rho_1(p) = {} & z_1[\exp(-c_2\frac{c_3-c_1}{c_2-c_3}p/T)-\frac{c_3-c_1}{c_2-c_3}z_3\exp(c_1p/T)\\ & +z_1\frac{c_1-c_2}{c_2-c_3}]/\{[\exp(-c_2\frac{c_3-c_1}{c_2-c_3}p/T)-z_1z_3]\\ & \times [\exp(c_1p/T)+z_1]\},\end{aligned}$$

$$\begin{aligned}2\pi\rho_3(q) = {} & z_3[\exp(-c_2\frac{c_3-c_1}{c_1-c_2}q/T)-\frac{c_3-c_1}{c_1-c_2}z_1\exp(c_3q/T)\\ & +z_3\frac{c_2-c_3}{c_1-c_3}]/\{[\exp(-c_2\frac{c_3-c_1}{c_1-c_2}q/T)-z_1z_3]\\ & \times [\exp(c_3q/T)+z_3]\}.\end{aligned} \tag{4.6}$$

This result is not derived by setting $g=0$ at first.

## §§4.3 Zero temperature limit: $T \to 0$

As $\varepsilon_1(p)$ and $\varepsilon_3(q)$ are monotonically increasing functions, there are certain Fermi levels $p_F$ and $q_F$;

$$\varepsilon_1(p) < 0, \quad \text{for } p<p_F, \qquad \varepsilon_1(p) > 0, \quad \text{for } p>p_F,$$

$$\varepsilon_1(p_F) = 0. \tag{4.7a}$$

Equation (3.19) gives

$$\rho_1(p) = 0, \quad \text{for } p>p_F, \qquad \rho_1^h(p) = 0, \quad \text{for } p<p_F, \tag{4.8a}$$

For $\varepsilon_3$ and $\rho_3$, the similar relation is valid, i.e.

$$\varepsilon_3(q) < 0, \quad \text{for } q<q_F, \qquad \varepsilon_3(q) > 0, \quad \text{for } q>q_F,$$

$$\varepsilon_3(q_F) = 0, \tag{4.7b}$$

$$\rho_3(q) = 0, \quad \text{for} \quad q>q_F, \qquad \rho_3^h(q) = 0, \quad \text{for } q<q_F. \tag{4.8b}$$

From eqs. (3.20) and (3.21), we obtain

$$\varepsilon_1(p) = -A_1T+c_1p-\frac{\kappa}{\pi(c_1-c_2)}\int_{q_K}^{q_F} dq\ K(p,q)\varepsilon_3(q),$$

$$\varepsilon_3(q) = -A_3T+c_3q-\frac{\kappa}{\pi(c_2-c_3)}\int_{p_K}^{p_F} dp\ K(p,q)\varepsilon_1(p), \tag{4.9}$$

$$2\pi\rho_1(p) = 1-\frac{2\kappa}{c_2-c_3}\int_{q_K}^{q_F} dq\ K(p,q)\rho_3(q),$$

$$2\pi\rho_3(q) = 1-\frac{2\kappa}{c_1-c_2}\int_{p_K}^{p_F} dp\ K(p,q)\rho_1(p). \tag{4.10}$$

## §6 Concluding Remarks

The Bethe state for the quantum three wave interaction models for three choices of statistics. The thermodynamics for the case with two kinds of fermions and one kind of bosons is studied. The main results are as follows.

1)In the study of the thermodynamics, the $Q_2$-particle does not appear explicitly. This means that the $Q_2$-particle is not fundamental and can be considered as a composite state of $Q_1$- and $Q_3$-particles.

2)The integral equations for the thermal equilibrium state are similar to that of nonlinear Schrödinger model with a repulsive interaction.[4)] The main differences are as follows. First the integral equations are for two fields coupled each other. Second we should introduce the cut-off momenta as the energy spectra for this model do not have lower bounds.

3)Three limiting cases are considered. In the zero temperature limit $T\to 0$, Fermi state appears. In the strong coupling limit $g\to\infty$, the $Q_1$- and $Q_3$-fields behave like free fermions. The result in the weak coupling limit $g \to 0$ each particle is not derived from the solution for $g=0$.

The themodynamics for Fermion model II is also studied quite similarly.

It is possible to study elementaly excitations. Two kinds of excitations; $Q_1$- and $Q_3$-excitations exist. We will publish the result in near future.

Recently Wadati and Sakagami showed the classical soliton is derived as a matrix element for an n-string in the limit $n \to \infty$ for Nonlinear Schrödinger model in attractive case.[6)] For Q3WI models, the same will be shown in the Boson model.

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